Probability Distributions and GAS Models for Realized Covariance Matrices

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Introduction

Objective: Model and forecast time series of Realized Covariance matrices (RCs).

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RCs are precise ex-post measurements of (co-)variation between financial assets based on high-frequency data.

They are important for portfolio risk minimization and derivative pricing.

For the modeling and forecasting we assume different probability distributions on the RCs and use so-called generalized autoregressive score (GAS) model dynamics.

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2. derivation of a new class of GAS models.
3. Derivation of own probability distributions with the desirable property of tail homogeneity (also with scores and Fisher information matrices).
4. Empirical fit and forecast comparison between the different distributions.
5. Illustration of relationships between the different distributions.

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Realized Covariance Matrices

Realized Covariance Matrix

$$\underset{(p\times p)}{\mathbf{R}_t} \equiv \sum_{j=1}^{m} \mathbf{r}_{j,t}\mathbf{r}_{j,t}^{\top}$$

- $\mathbf{r}_{j,t}$ is the j'th intraday return vector $(p \times 1)$ on day t.
- $\mathbf{R}_t$ is symmetric positive definite (s.p.d.) (if $m \geq p$).
- $\mathbf{R}_t$ is a consistent estimator of the integrated covariance of day t.

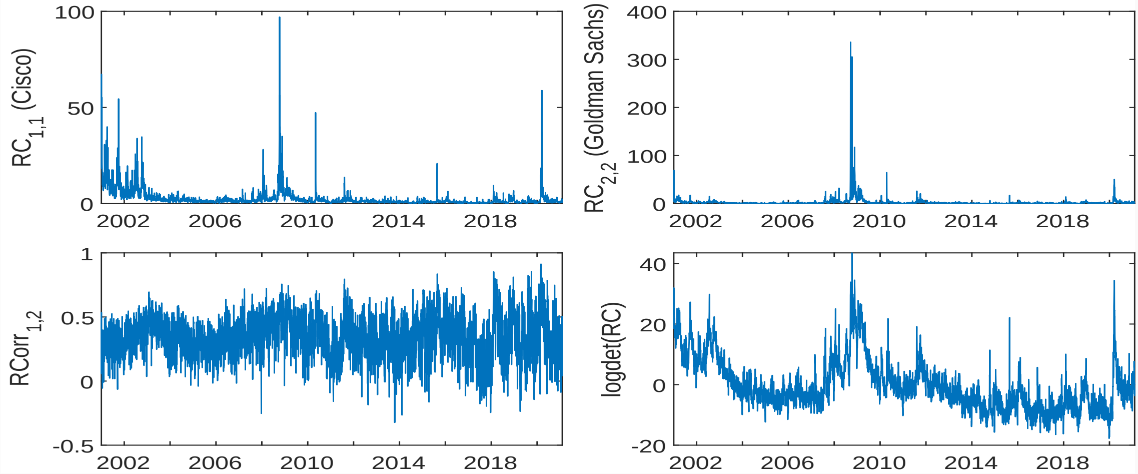
Dataset

From *Quantquote* we obtained 1-min OHLC equity data from which we constructed trading-daily RCs created from

- 5min returns with subsampling over the trading day
- for a random selection two of 5, 10, 25 and 50
- from 02.01.2001 to 05.02.2021.

Data Plot

10 assets dataset: csco,gs,amat,bac,c,intc,jnj,jnpr,jpm,ms



Probability Distributions

Probability Distributions for RCs - Setup

$$\mathbf{R}_t|\mathcal{F}_{t-1} \sim d(\mathbf{\Sigma}_t, \boldsymbol{\theta})$$

- d is one of the considered probability distributions.

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- $\underset{(p\times p)}{\mathbf{\Sigma}_t}$ is s.p.d. and time varying parameter matrix with $\mathbb{E}\left[\mathbf{R}_t|\mathcal{F}_{t-1}\right] = \mathbf{\Sigma}_t$,
- $\boldsymbol{\theta} = (\mathbf{n}, \boldsymbol{\nu})$ collects the so called degree of freedom parameters, which are depending on the chosen distribution d, both either $p \times 1$ or scalars. $\boldsymbol{\nu}$ is only present in some distributions.

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- All distributions considered are based either on

$$\underline{\mathbf{B}} = \begin{bmatrix} \sqrt{\chi^2_{n_1-1+1}} & 0 & \dots & 0 \\ \mathcal{N}(0,1) & \ddots & 0 & \vdots \\ \vdots & \mathcal{N}(0,1) & \ddots & 0 \\ \mathcal{N}(0,1) & \dots & \mathcal{N}(0,1) & \sqrt{\chi^2_{n_p-p+1}} \end{bmatrix} \text{ or } \bar{\mathbf{B}} = \begin{bmatrix} \sqrt{\chi^2_{\nu_1-p+1}} & \mathcal{N}(0,1) & \dots & \mathcal{N}(0,1) \\ 0 & \ddots & \mathcal{N}(0,1) & \vdots \\ \vdots & 0 & \ddots & \mathcal{N}(0,1) \\ 0 & \dots & 0 & \sqrt{\chi^2_{\nu_p-1+1}}, \end{bmatrix}$$

or a combination of both.

Probability Distributions used in Literature

Wishart: Too thin-tailed!

Golosnoy et al. (J.Econom., 2012); Gorgi et al. (J.F.Econom., 2019)

Non-central-Wishart: Infeasible for $p > 3$.

Yu et al. (JBES, 2017)

Riesz: Heterogeneous liquidity interpretation, more flexible, but also too thin-tailed.

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Inverse Riesz and **F-Riesz**: Fat-tailed and flexible, F-Riesz: Tail-heterogeneity.

Blasques, et al. (WP, 2021)

Probability Distributions - New t-named Family

The stochastic representation of inverse t-Riesz distribution and the F-Riesz distribution are given by,

$$\Gamma_{\left(\frac{n}{2},\frac{2}{n}\right)} \mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}} \bar{\mathbf{B}}^{-\top} \bar{\mathbf{B}}^{-1} \mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}}$$

and

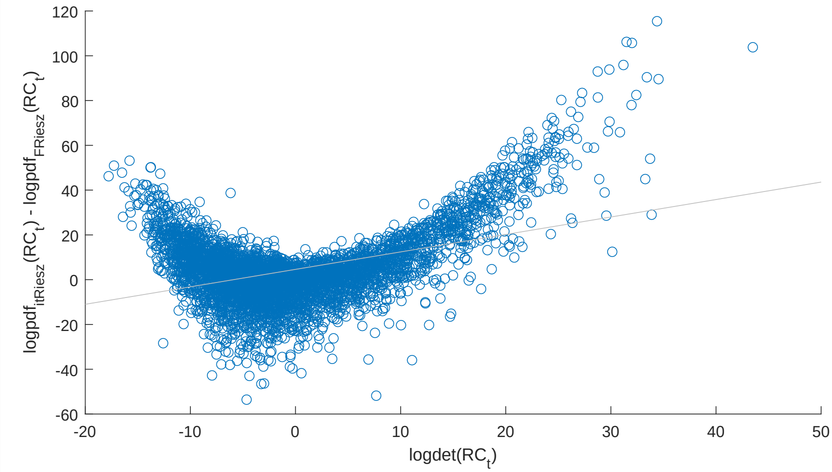
$$\mathrm{dg}(\mathbf{m}^{\mathcal{F}\mathcal{R}'})^{-\frac{1}{2}} \bar{\mathbf{B}}^{-\top} \underline{\mathbf{B}} \underline{\mathbf{B}}^{\top} \bar{\mathbf{B}}^{-1} \mathrm{dg}(\mathbf{m}^{\mathcal{F}\mathcal{R}'})^{-\frac{1}{2}},$$

respectively. Given $\bar{\mathbf{B}}$,

- a tail realization of the gamma distribution yields a tail realization of the inverse t-Riesz distribution (tail-homogeneity), whereas
- a tail realization of one of the entries on the main diagonal of $\underline{\mathbf{B}}$, does not necessarily yield a tail realization of the F-Riesz distribution, since other realizations on the main diagonal might not lie in the tail (tail-heterogeneity).

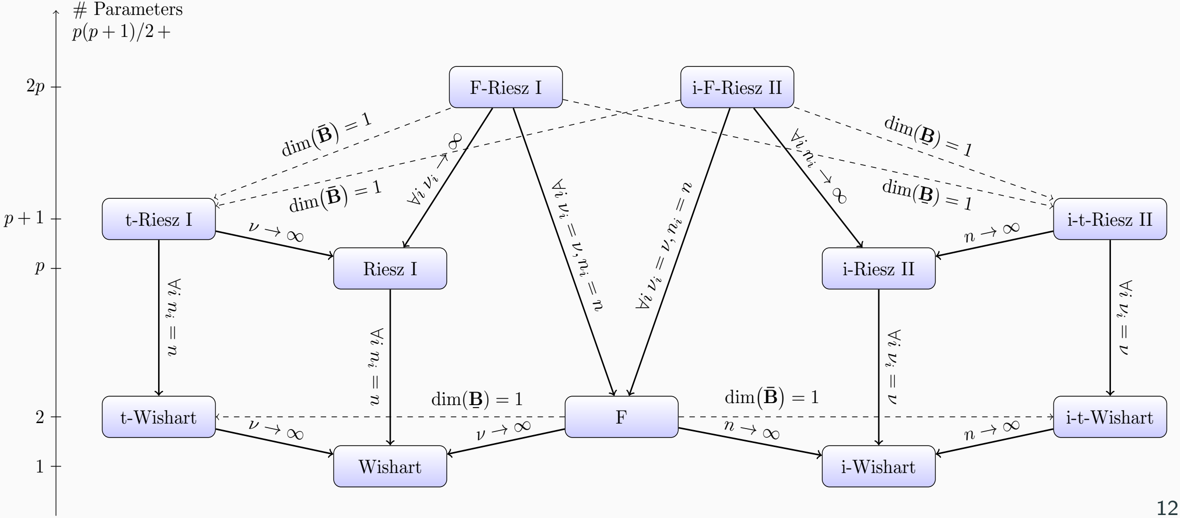
Static Probability Distribution - Tail Homogeneity vs Tail Heterogeneity

10 asset dataset



Probability Distribution Relationships

Stochastic Representations
Probability Density Functions



Generalized Autoregressive Score (GAS) Models

Generalized Autoregressive Score (GAS) Models - Theory Slide 1/2

$$\mathbf{R}_t|\mathcal{F}_{t-1} \sim d(\boldsymbol{\Sigma}_t, \boldsymbol{\theta})$$
$$\boldsymbol{\Sigma}_t = (1-\beta)\boldsymbol{\Xi} + \alpha \mathbf{S}_{t-1} + \beta \boldsymbol{\Sigma}_{t-1},$$

α and β are scalars.

$$\mathbf{S}_t = \mathrm{ivech}\left(\mathcal{I}_t^{-1} \nabla_t^\top\right),$$

where

$$\nabla_t = \frac{\partial log\ p_d(\mathbf{R}_t|\boldsymbol{\Sigma}_t, \boldsymbol{\theta}; \mathcal{F}_{t-1})}{\partial \mathrm{vech}\left(\boldsymbol{\Sigma}_t\right)^\top} \quad \text{and} \quad \mathcal{I}_t = -\mathbb{E}\left[\frac{\partial^2 log\ p_d(\mathbf{R}_t|\boldsymbol{\Sigma}_t, \boldsymbol{\theta}; \mathcal{F}_{t-1})}{\partial \mathrm{vech}\left(\boldsymbol{\Sigma}_t\right) \partial \mathrm{vech}\left(\boldsymbol{\Sigma}_t\right)^\top}\right].$$

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Contribution: Derivation of ∇_t (⏩ ∇_t) for all distributions and $\mathcal{I}_t$ (⏩ $\mathcal{I}_t$) for all except the (inverse) F-Riesz.

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Contribution: Derivation of ∇_t (∇_t) for all distributions and $\mathcal{I}_t$ ($\mathcal{I}_t$) for all except the (inverse) F-Riesz.

$\mathcal{I}_t^{-1}$ **requires multiplications and inversions of** $p^2 \times p^2$ **matrices!**

Generalized Autoregressive Score (GAS) Models - Theory Slide 2/2

If, for any Riesz-named distribution we use $\mathcal{I}_t$ of its Wishart-based counterpart instead of its own,[1] then

$$\boldsymbol{\Sigma}_t = (1-\beta)\boldsymbol{\Xi} + \alpha\left(\frac{\alpha_{\boldsymbol{\theta}}}{2}\boldsymbol{\Sigma}_t\left(\triangle_t + \triangle_t^\top\right)\boldsymbol{\Sigma}_t + \beta_{\boldsymbol{\theta}}\,\mathrm{tr}\left(\boldsymbol{\Sigma}_t\triangle_t\right)\boldsymbol{\Sigma}_t\right) + \beta\boldsymbol{\Sigma}_{t-1} \tag{1}$$

where $\triangle_t$ is the score matrix w.r.t. $\boldsymbol{\Sigma}_t$, ignoring symmetry and $\alpha_{\boldsymbol{\theta}}$ and $\beta_{\boldsymbol{\theta}}$ depend only on the degree of freedom parameters of the respective distribution.

[1] By setting the degree(s) of freedom equal to the average of the corresponding degree of freedom parameter vector(s). Riesz Example

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Final Model:

$$\boldsymbol{\Sigma}_t = (1-\beta)\boldsymbol{\Xi} + \alpha_1\boldsymbol{\Sigma}_t\left(\triangle_t + \triangle_t^\top\right)\boldsymbol{\Sigma}_t + \alpha_2\,\mathrm{tr}\left(\boldsymbol{\Sigma}_t\triangle_t\right)\boldsymbol{\Sigma}_t + \beta\boldsymbol{\Sigma}_{t-1} \tag{2}$$

[1] By setting the degree(s) of freedom equal to the average of the corresponding degree of freedom parameter vector(s). Riesz Example

Generalized Autoregressive Score (GAS) Models - Estimation

Final Model:

$$\mathbf{R}_t|\mathcal{F}_{t-1} \sim d(\boldsymbol{\Sigma}_t, \boldsymbol{\theta}) \tag{3}$$

$$\boldsymbol{\Sigma}_t = (1-\beta)\boldsymbol{\Xi} + \alpha_1 \boldsymbol{\Sigma}_t \left(\triangle_t + \triangle_t^\top\right) \boldsymbol{\Sigma}_t + \alpha_2 \operatorname{tr}\left(\boldsymbol{\Sigma}_t \triangle_t\right) \boldsymbol{\Sigma}_t + \beta \boldsymbol{\Sigma}_{t-1} \tag{4}$$

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We have $\mathbb{E}\left[\mathbf{R}_t\right] = \mathbb{E}\left[\boldsymbol{\Sigma}_t\right] = \boldsymbol{\Xi}$, such that we can

1. estimate $\hat{\boldsymbol{\Xi}} = \frac{1}{T}\sum_{t=1}^{T} \mathbf{R}_t$ and then
2. estimate α_1, α_2, β and $\boldsymbol{\theta}$ via standard numerical maximum likelihood.

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For the Riesz-named distributions $\#param > p$, such that for $p > 50$, estimation becomes infeasible. For the Wishart-based distributions it is feasible for vast p.

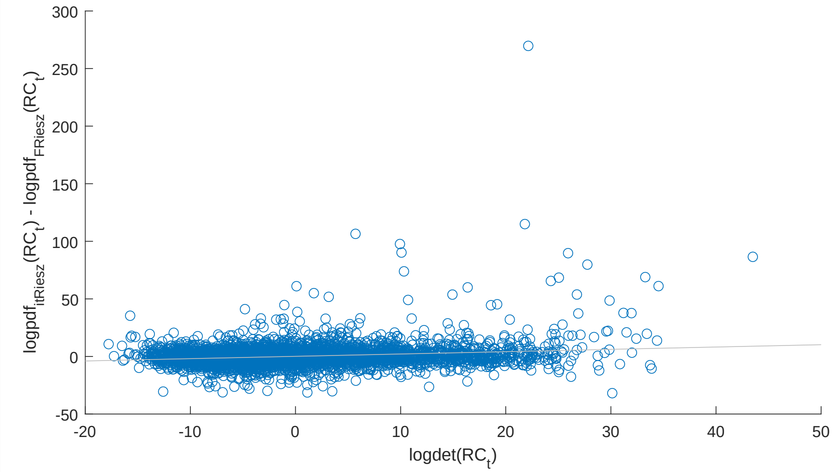
GAS Model Fit - Log Likelihood Values

# Assets:	5	5	10	10	25	25
Wishart	-29555	-25819	-34377	-38397	140907	
Riesz	-24713	-22815	-18507	-20851	245176	
iWishart	-10729	-10588	34724	30197	608212	
iRiesz	-8850	-9246	39021	35790	632471	
tWishart	-15604	-14160	4047	-2067	273354	
tRiesz	-13401	-12252	14258	9117	352243	
itWishart	-6691	-6162	49386	44115	659591	
itRiesz	-5181	-4954	53406	48993	680302	
F	-10409	-10418	34904	30626	611107	
FRiesz	-4778	-4516	52417	50165	683462	
iFRiesz	-6596	-6527	46209	45236	663824	

Estimated Parameters

GAS Model Fit - Tail Homogeneity vs Tail Heterogeneity

10 asset dataset



GAS Model Forecasting Ability - Setting

We use the

- 10 assets dataset and
- starting from 02.01.2007, estimate the model daily with 1250 trailing observations,
- make 1-step ahead forecasts
- and evaluate the forecasting ability with mse, log-score and gmvp variance.

⏩ data plot

GAS Model Forecasting Ability - MSE

$\sum_t \sum_{i\leq j}(\mathbf{R}_{ij,t+1} - \widehat{\boldsymbol{\Sigma}}_{ij,t+1})^2$

	Mean Squared Error			
# Assets:	5	5	10	10
Wishart	506	773	853	215
Riesz	516	778	844	240
iWishart	519	791	853	252
iRiesz	521	797	859	259
F	512	789	843	241
FRiesz	523	807	896	266
iFRiesz	510	794	840	254
tWishart	504	785	823	226
tRiesz	503	782	822	259
itWishart	498	767	814	219
itRiesz	500	769	813	224

GAS Model Forecasting Ability - Log Likelihood Loss (a.k.a. Log Score)

Plug $\mathbf{R}_{t+1}$ into the in time-t forecasted log probability density function,

$$\mathit{log}\ p_d(\mathbf{R}_{t+1}|\widehat{\boldsymbol{\Sigma}_{t+1}}, \widehat{\boldsymbol{\theta}}; \mathcal{F}_t).$$

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	- Log-Score			
# Assets:	5	5	10	10
Wishart	9.2	8.5	-9.2	-11.8
Riesz	8.0	7.9	-12.4	-14.6
iWishart	5.3	5.3	-23.6	-24.6
iRiesz	4.8	5.1	-24.5	-25.6
F	5.1	5.2	-23.7	-24.7
FRiesz	3.9	4.0	-27.5	-28.8
iFRiesz	4.2	4.4	-26.2	-28.0
tWishart	5.9	5.7	-18.7	-19.8
tRiesz	5.4	5.4	-20.6	-21.9
itWishart	4.2	4.2	-27.1	-27.9
itRiesz	3.8	3.9	-27.9	-28.7

Forecasting 02.01.2007 - 05.02.2021.

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# Assets:	5	5	10	10
Wishart	9.2	8.5	-9.2	-11.8
Riesz	8.0	7.9	-12.4	-14.6
iWishart	5.3	5.3	-23.6	-24.6
iRiesz	4.8	5.1	-24.5	-25.6
F	5.1	5.2	-23.7	-24.7
FRiesz	3.9	4.0	-27.5	-28.8
iFRiesz	4.2	4.4	-26.2	-28.0
tWishart	5.9	5.7	-18.7	-19.8
tRiesz	5.4	5.4	-20.6	-21.9
itWishart	4.2	4.2	-27.1	-27.9
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Forecasting 02.01.2007 - 05.02.2021.

	- Log-Score			
# Assets:	5	5	10	10
Wishart	18.0	18.0	30.7	15.6
Riesz	17.3	17.6	26.6	12.7
iWishart	15.5	15.3	15.4	2.7
iRiesz	14.7	15.1	14.7	1.6
F	15.0	15.3	15.3	2.5
FRiesz	13.8	13.9	11.2	-1.2
iFRiesz	14.1	14.4	12.6	-0.3
tWishart	15.3	15.4	19.0	6.7
tRiesz	14.9	15.2	17.7	5.0
itWishart	13.8	14.1	11.1	-1.1
itRiesz	13.5	13.9	10.4	-2.0

Forecasting 02.01.2007 - 31.12.2010.

Economic Relevance - GAS Model Forecasting - GMVP

Investors are interested in **minimizing** their expected portfolio variance. For a given set of assets they can do so by choosing the optimal portfolio weights $\mathbf{w}_t$.

$$\min_{\mathbf{w}_t} \mathbf{w}_t^\top \boldsymbol{\Sigma}_t \mathbf{w}_t$$

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$$\underset{\mathbf{w}_t}{min}\, \mathbf{w}_t^\top \boldsymbol{\Sigma}_t \mathbf{w}_t$$

The optimal weights then are

$$\mathbf{w}_t^* = \frac{\boldsymbol{\Sigma}_t^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}_t^{-1}\mathbf{1}}.$$

GAS Model Forecasting Ability - GMVP

1. Given predictions $\mathbf{\Sigma}_{t+1}$, get optimal $t+1$ weights $\mathbf{w}^*_{t+1}$,
2. then calculated the actually realized portfolio variance $\left(\mathbf{w}^*_{t+1}\right)^\top \mathbf{R}_{t+1}\,\mathbf{w}^*_{t+1}$.

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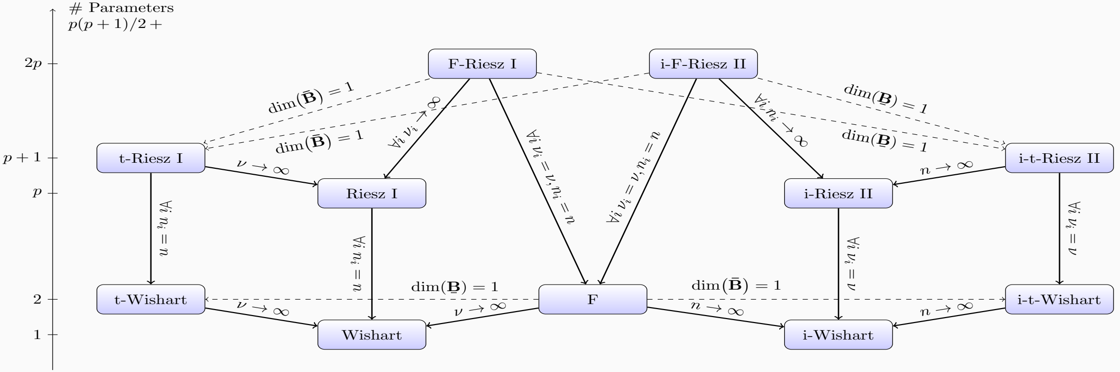
	GMVP Variances			
# Assets:	5	5	10	10
Wishart	0.895	1.124	0.585	0.610
Riesz	0.903	1.133	0.598	0.627
iWishart	0.950	1.154	0.597	0.623
iRiesz	0.907	1.158	0.607	0.630
F	0.901	1.143	0.595	0.619
FRiesz	0.915	1.167	0.607	0.637
iFRiesz	0.900	1.150	0.611	0.636
tWishart	0.893	1.125	0.585	0.615
tRiesz	0.900	1.135	0.612	0.626
itWishart	0.893	1.134	0.591	0.616
itRiesz	0.894	1.137	0.593	0.623

Conclusions

Conclusions

- The newly proposed GAS models do a good job at exposing fit and forecasting differences among different distributions.
- All probability distributions so far used in the literature are closely related to each other.
- Fat tailed distributions fit better and forecast better than thin tailed ones.
- The newly proposed "t-named" distributions perform amongst the best in terms of fit and forecasting ability and
- their tail homogeneity is advantageous in times of market wide crises.

Probability Distribution Relationships



Stochastic Representations

Wishart	$\frac{1}{n}\underline{\mathcal{B}}\underline{\mathcal{B}}^\top$	Riesz I	$\mathrm{dg}(\mathbf{n})^{-\frac{1}{2}}\underline{\mathbf{B}}\underline{\mathbf{B}}^\top\mathrm{dg}(\mathbf{n})^{-\frac{1}{2}}$
Inverse Wishart	$(n-p-1)\bar{\mathcal{B}}^{-\top}\bar{\mathcal{B}}^{-1}$	Inverse Riesz II	$\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}}\bar{\mathbf{B}}^{-\top}\bar{\mathbf{B}}^{-1}\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}}$
t-Wishart	$\frac{\nu-2}{\nu n}\Gamma^{-1}_{(\frac{\nu}{2},\frac{2}{\nu})}\underline{\mathcal{B}}\underline{\mathcal{B}}^\top$	t-Riesz I	$\frac{\nu-2}{\nu}\Gamma^{-1}_{(\frac{\nu}{2},\frac{2}{\nu})}\mathrm{dg}(\mathbf{n})^{-\frac{1}{2}}\underline{\mathbf{B}}\underline{\mathbf{B}}^\top\mathrm{dg}(\mathbf{n})^{-\frac{1}{2}}$
Inverse t-Wishart	$(n-p-1)\Gamma_{(\frac{n}{2},\frac{2}{n})}\bar{\mathcal{B}}^{-\top}\bar{\mathcal{B}}^{-1}$	Inverse t-Riesz II	$\Gamma_{(\frac{n}{2},\frac{2}{n})}\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}}\bar{\mathbf{B}}^{-\top}\bar{\mathbf{B}}^{-1}\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-\frac{1}{2}}$
F	$\frac{\nu-p-1}{n}\bar{\mathcal{B}}^{-\top}\underline{\mathcal{B}}\underline{\mathcal{B}}^\top\bar{\mathcal{B}}^{-1}$	F-Riesz I	$\mathrm{dg}(\mathbf{m}^{\mathcal{FR}'})^{-\frac{1}{2}}\bar{\mathbf{B}}^{-\top}\underline{\mathbf{B}}\underline{\mathbf{B}}^\top\bar{\mathbf{B}}^{-1}\mathrm{dg}(\mathbf{m}^{\mathcal{FR}'})^{-\frac{1}{2}}$
F	$\frac{\nu-p-1}{n}\underline{\mathcal{B}}\bar{\mathcal{B}}^{-\top}\bar{\mathcal{B}}^{-1}\underline{\mathcal{B}}^\top$	Inverse F-Riesz II	$\mathrm{dg}(\mathbf{m}^{i\mathcal{FR}''})^{-\frac{1}{2}}\underline{\mathbf{B}}\bar{\mathbf{B}}^{-\top}\bar{\mathbf{B}}^{-1}\underline{\mathbf{B}}^\top\mathrm{dg}(\mathbf{m}^{i\mathcal{FR}''})^{-\frac{1}{2}}$

Probability Density Functions 1/2

Distribution	Probability Density Function, $p(\mathbf{R}\mid\boldsymbol{\Sigma},\boldsymbol{\theta})$			
Wishart	$\frac{n^{np/2}}{2^{np/2}}\frac{1}{\Gamma_p(n/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert^{\frac{n}{2}}$	$\operatorname{etr}\left(-\frac{1}{2}n\mathbf{Z}\right)$
Riesz	$\frac{\prod_{i=1}^{p} n_i^{n_i/2}}{2^{p\bar{\mathbf{n}}/2}}\frac{1}{\Gamma_p(\mathbf{n}/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{\frac{\mathbf{n}}{2}}$	$\operatorname{etr}\left(-\frac{1}{2}\operatorname{dg}(\mathbf{n})\,\mathbf{Z}\right)$
Inverse Wishart	$\frac{(\nu-p-1)^{\nu p/2}}{2^{\nu p/2}}\frac{1}{\Gamma_p(\nu/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert^{-\frac{\nu}{2}}$	$\operatorname{etr}\left(-\frac{1}{2}(\nu-p-1)\mathbf{Z}^{-1}\right)$
Inverse Riesz	$\frac{\prod_{i=1}^{p} m_i^{-\nu_i/2}}{2^{p\bar{\boldsymbol{\nu}}/2}}\frac{1}{\Gamma_p(\overleftarrow{\boldsymbol{\nu}}/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{-\frac{\boldsymbol{\nu}}{2}}$	$\operatorname{etr}\left(-\frac{1}{2}\operatorname{dg}(\mathbf{m})^{-1}\,\mathbf{Z}^{-1}\right)$
t-Wishart	$\left(\frac{n}{\nu-2}\right)^{pn/2}\frac{\Gamma((\nu+pn)/2)}{\Gamma_p(n/2)\Gamma(\nu/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert^{\frac{n}{2}}$	$\left(1+\frac{n}{\nu-2}\operatorname{tr}(\mathbf{Z})\right)^{-\frac{\nu+pn}{2}}$
t-Riesz	$\frac{\prod_{i=1}^{p} n_i^{n_i/2}}{(\nu-2)^{p\bar{\mathbf{n}}/2}}\frac{\Gamma((\nu+p\bar{\mathbf{n}})/2)}{\Gamma_p(\mathbf{n}/2)\Gamma(\nu/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{\frac{\mathbf{n}}{2}}$	$\left(1+\frac{1}{\nu-2}\operatorname{tr}\left(\operatorname{dg}(\mathbf{n})\mathbf{Z}\right)\right)^{-\frac{\nu+p\bar{\mathbf{n}}}{2}}$

Table 1: Probability density functions of all considered distributions. Recall that $\mathbf{Z}=\mathbf{C}^{-1}\mathbf{R}\mathbf{C}^{-\top}$, where $\mathbf{C}$ is the lower Cholesky factor of $\boldsymbol{\Sigma}$. A bar on top of a vector denotes the average of its entries, e.g. $\bar{\mathbf{n}}=p^{-1}\sum_{i=1}^{p} n_i$, left arrow on top of a vector denotes the original vector in reverse order, e.g. $\overleftarrow{\mathbf{n}}=(n_p,n_{p-1},\ldots,n_1)^\top$.

Probability Density Functions 2/2

Distribution	Probability Density Function, $p(\mathbf{R}\vert\mathbf{\Sigma},\boldsymbol{\theta})$			
Inverse t-Wishart	$\left(\frac{\nu-p-1}{n}\right)^{\nu p/2}\frac{\Gamma((n+p\nu)/2)}{\Gamma_p(\nu/2)\Gamma(n/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert^{-\frac{\nu}{2}}$	$\left(1+\frac{\nu-p-1}{n}\mathrm{tr}(\mathbf{Z}^{-1})\right)^{-\frac{n+p\nu}{2}}$
Inverse t-Riesz	$\frac{\prod_{i=1}^{p}(m_i^{i\mathcal{R}''})^{-\nu_i/2}}{n^{p\bar{\boldsymbol{\nu}}/2}}\frac{\Gamma((n+p\bar{\boldsymbol{\nu}})/2)}{\Gamma_p(\overleftarrow{\boldsymbol{\nu}}/2)\Gamma(n/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{-\frac{\boldsymbol{\nu}}{2}}$	$\left(1+\frac{1}{n}\mathrm{tr}\left(\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-1}\mathbf{Z}^{-1}\right)\right)^{-\frac{n+p\bar{\boldsymbol{\nu}}}{2}}$
F	$\left(\frac{n}{\nu-p-1}\right)^{np/2}\frac{\Gamma_p((\nu+n)/2)}{\Gamma_p(n/2)\Gamma_p(\nu/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert^{\frac{n}{2}}$	$\left\lvert\mathbf{I}+\frac{n}{\nu-p-1}\mathbf{Z}\right\rvert^{-\frac{\nu+n}{2}}$
F-Riesz	$\prod_{i=1}^{p}(m_i^{\mathcal{FR}'})^{n_i/2}\frac{\Gamma_p\left((\overleftarrow{\mathbf{n}}+\overleftarrow{\boldsymbol{\nu}})/2\right)}{\Gamma_p(\mathbf{n}/2)\Gamma_p(\overleftarrow{\boldsymbol{\nu}}/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{\frac{\mathbf{n}}{2}}$	$\lvert\mathbf{I}+\mathrm{dg}(\mathbf{m}^{\mathcal{FR}'})^{1/2}\mathbf{Z}\mathrm{dg}(\mathbf{m}^{\mathcal{FR}'})^{1/2}\rvert_{-\frac{\mathbf{n}+\boldsymbol{\nu}}{2}}$
Inverse F-Riesz	$\prod_{i=1}^{p}\left(m_i^{i\mathcal{R}''}\right)^{-\nu_i/2}\frac{\Gamma_p((\boldsymbol{\nu}+\mathbf{n})/2)}{\Gamma_p(\overleftarrow{\boldsymbol{\nu}}/2)\Gamma_p(\mathbf{n}/2)}$	$\lvert\mathbf{R}\rvert^{-\frac{p+1}{2}}$	$\lvert\mathbf{Z}\rvert_{-\frac{\boldsymbol{\nu}}{2}}$	$\left\lvert\left(\mathbf{I}+\mathrm{dg}\left(\mathbf{m}^{i\mathcal{R}''}\right)^{-\frac{1}{2}}\mathbf{Z}^{-1}\mathrm{dg}\left(\mathbf{m}^{i\mathcal{R}''}\right)^{-\frac{1}{2}}\right)^{-1}\right\rvert_{\frac{\boldsymbol{\nu}+}{2}}$

Table 2: Probability density functions of all considered distributions. Recall that $\mathbf{Z}=\mathbf{C}^{-1}\mathbf{R}\mathbf{C}^{-\top}$, where $\mathbf{C}$ is the lower Cholesky factor of $\mathbf{\Sigma}$. A bar on top of a vector denotes the average of its entries, e.g. $\bar{\mathbf{n}}=p^{-1}\sum_{i=1}^{p}n_i$, left arrow on top of a vector denotes the original vector in reverse order, e.g. $\overleftarrow{\mathbf{n}}=(n_p,n_{p-1},\ldots,n_1)^\top$.

Scores

Distribution	Score, $\nabla = \mathbf{G}^\top \mathrm{vec}\,(\triangle)$
Wishart	$\frac{1}{2}\mathbf{G}^\top \mathrm{vec}\left(n\,\boldsymbol{\Sigma}^{-1}\mathbf{R}\boldsymbol{\Sigma}^{-1} - n\,\boldsymbol{\Sigma}^{-1}\right)$
Inverse Wishart	$-\frac{1}{2}\mathbf{G}^\top \mathrm{vec}\left((\nu - p - 1)\mathbf{R}^{-1} - \nu\boldsymbol{\Sigma}^{-1}\right)$
t-Wishart	$\frac{1}{2}\mathbf{G}^\top \mathrm{vec}\left(n\,\frac{\nu+pn}{\nu-2+n\,\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{R})}\boldsymbol{\Sigma}^{-1}\mathbf{R}\boldsymbol{\Sigma}^{-1} - n\,\boldsymbol{\Sigma}^{-1}\right)$
Inverse t-Wishart	$-\frac{1}{2}\mathbf{G}^\top \mathrm{vec}\left(\frac{(n+p\nu)(\nu-p-1)}{n+(\nu-p-1)\,\mathrm{tr}(\boldsymbol{\Sigma}\mathbf{R}^{-1})}\mathbf{R}^{-1} - \nu\boldsymbol{\Sigma}^{-1}\right)$
F	$-\frac{1}{2}\mathbf{G}^\top \mathrm{vec}\left((n+\nu)\left(\boldsymbol{\Sigma} + \frac{n}{\nu-p-1}\mathbf{R}\right)^{-1} - \nu\boldsymbol{\Sigma}^{-1}\right)$
Riesz	$\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\mathbf{C}^{-\top}\mathrm{dg}\,(\mathbf{n})\,\mathbf{Z} - \mathbf{C}^{-\top}\mathrm{dg}(\mathbf{n})\right)\right)\mathbf{C}^{-1}\right)$
Inverse Riesz	$-\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\mathbf{R}^{-1}\mathbf{C}\,\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-1} - \mathbf{C}^{-\top}\mathrm{dg}(\boldsymbol{\nu})\right)\right)\mathbf{C}^{-1}\right)$
t-Riesz	$\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\frac{\nu+p\bar{\mathbf{n}}}{\nu-2+\mathrm{tr}(\mathrm{dg}(\mathbf{n})\mathbf{Z})}\mathbf{C}^{-\top}\mathrm{dg}\,(\mathbf{n})\,\mathbf{Z} - \mathbf{C}^{-\top}\mathrm{dg}(\mathbf{n})\right)\right)\mathbf{C}^{-1}\right)$
Inverse t-Riesz	$-\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\frac{n+p\bar{\boldsymbol{\nu}}}{n+\,\mathrm{tr}(\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})^{-1}\mathbf{Z}-1)}\mathbf{R}^{-1}\mathbf{C}\mathrm{dg}\left(\mathbf{m}^{i\mathcal{R}''}\right)^{-1} - \mathbf{C}^{-\top}\mathrm{dg}\,(\boldsymbol{\nu})\right)\right)\mathbf{C}^{-1}\right)$
F-Riesz	$\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\mathbf{C}^{-\top}\mathrm{dg}(\boldsymbol{\nu}) - \mathbf{C}_{\mathbf{B}}^{-\top}\mathrm{dg}(\boldsymbol{\nu}+\mathbf{n})\mathbf{C}_{\mathbf{B}}^{-1}\mathbf{C}\mathrm{dg}(\mathbf{m}^{\mathcal{F}\mathcal{R}'})^{-1}\right)\right)\mathbf{C}^{-1}\right)$
Inverse F-Riesz	$-\mathbf{G}^\top \mathrm{vec}\left(\mathbf{C}^{-\top}\Phi\left(\mathbf{C}^\top \mathrm{tril}\left(\mathbf{C}^{-\top}\mathrm{dg}\,(\mathbf{n}) - \mathbf{C}^{-\top}\mathrm{dg}(\mathbf{m}^{i\mathcal{R}''})\mathbf{C}^{-1}\mathbf{C}_{\mathbf{B}_2}\mathrm{dg}\,(\mathbf{n}+\boldsymbol{\nu})\,\mathbf{C}_{\mathbf{B}_2}^\top\mathbf{C}^{-\top}\right)\right)\mathbf{C}^{-1}\right)$

Fisher Information Matrices

Distribution	Fisher Information Matrix $\mathcal{I}$
Wishart	$\frac{n}{2}\mathbf{G}^\top \left(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}\right) \mathbf{G}$
Inverse Wishart	$-\frac{\nu}{2}\mathbf{G}^\top \left(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}\right) \mathbf{G}$
t-Wishart	$\frac{n}{2}\mathbf{G}^\top \left(\frac{\nu+pn}{\nu+pn+2} \left(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}\right) - \frac{n}{(\nu+pn+2)} \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right) \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right)^\top\right) \mathbf{G}$
Inverse t-Wishart	$-\frac{\nu}{2}\mathbf{G}^\top \left(\frac{n+p\nu}{n+p\nu+2} \left(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}\right) - \frac{\nu}{(n+p\nu+2)} \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right) \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right)^\top\right) \mathbf{G}$
F	$\frac{1}{2}\mathbf{G}^\top \left(\left(\nu + (n+\nu)(c_3+c_4)\right) \left(\mathbf{\Sigma}^{-1} \otimes \mathbf{\Sigma}^{-1}\right) + (n+\nu)c_4 \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right) \mathrm{vec}\left(\mathbf{\Sigma}^{-1}\right)^\top\right) \mathbf{G}$

Table 4: Fisher information matrices of all considered Wishart-based distributions. **G** denotes the duplication matrix.

Fisher Information Matrices - Riesz

$\boldsymbol{\Omega} = \mathbf{C}\mathrm{dg}(\mathbf{n})^{-1}\mathbf{C}^\top.$

$$\mathcal{I} = \left(\frac{\partial \mathrm{vech}(\boldsymbol{\Omega})}{\partial \mathrm{vech}(\boldsymbol{\Sigma})^\top}\right)^\top \mathcal{I}_{\boldsymbol{\Omega}} \frac{\partial \mathrm{vech}(\boldsymbol{\Omega})}{\partial \mathrm{vech}(\boldsymbol{\Sigma})^\top}, \tag{5}$$

with

$$\mathcal{I}_{\boldsymbol{\Omega}} = \frac{1}{2}\mathbf{G}^\top \left(\boldsymbol{\Omega}^{-1} \otimes \boldsymbol{\Omega}^{-1}\right) \left(\mathbf{C}_{\boldsymbol{\Omega}}\, \mathrm{dg}(\mathbf{n}) \otimes \mathbf{I}\right) \mathbf{F}^\top \left(\mathbf{G}^{+} \left(\mathbf{C}_{\boldsymbol{\Omega}} \otimes \mathbf{I}\right) \mathbf{F}^\top\right)^{-1}. \tag{6}$$

For $\mathbf{n} = (n, n, \ldots, n)$ this reduce to

$$\frac{n}{2}\mathbf{G}^\top \left(\boldsymbol{\Sigma}^{-1} \otimes \boldsymbol{\Sigma}^{-1}\right). \tag{7}$$

GAS Models - Parameter Estimates - α_1

# Assets:	5	5	10	10	25	25
$10^{-2}\times$						
Wishart	1.446	1.105	0.600	0.609	0.106	
Riesz	0.740	0.938	0.346	0.315	0.079	
iWishart	0.443	0.611	0.336	0.342	0.127	
iRiesz	0.472	0.541	0.293	0.293	0.113	
tWishart	0.699	0.719	0.375	0.426	0.115	
tRiesz	0.549	0.607	0.264	0.270	0.085	
itWishart	0.594	0.639	0.351	0.371	0.138	
itRiesz	0.517	0.557	0.314	0.320	0.122	
F	0.644	0.715	0.365	0.393	0.146	
FRiesz	0.367	0.330	0.211	0.232	0.103	
iFRiesz	0.501	0.459	0.272	0.261	0.113	

GAS Models - Parameter Estimates - α_2

# Assets:	5	5	10	10	25	25
$10^{-2}\times$						
Wishart	0.554	0.690	0.332	0.279	0.073	
Riesz	0.753	0.810	0.438	0.325	0.089	
iWishart	0.534	0.518	0.250	0.229	0.068	
iRiesz	0.644	0.577	0.275	0.253	0.077	
tWishart	5.090	4.796	4.516	4.693	4.479	
tRiesz	4.977	4.956	4.657	4.411	4.321	
itWishart	2.942	3.357	2.727	2.427	1.606	
itRiesz	3.049	3.579	2.919	2.537	1.669	
F	0.605	0.604	0.262	0.249	0.076	
FRiesz	1.080	1.175	0.472	0.449	0.127	
iFRiesz	1.012	1.101	0.414	0.410	0.112	

GAS Models - Parameter Estimates - β

# Assets:	5	5	10	10	25	25
Wishart	0.9832	0.9880	0.9926	0.9848	0.9969	
Riesz	0.9926	0.9882	0.9957	0.9924	0.9980	
iWishart	0.9969	0.9945	0.9968	0.9939	0.9974	
iRiesz	0.9969	0.9956	0.9974	0.9952	0.9980	
tWishart	0.9946	0.9940	0.9966	0.9915	0.9972	
tRiesz	0.9967	0.9950	0.9978	0.9953	0.9982	
itWishart	0.9956	0.9942	0.9968	0.9932	0.9972	
itRiesz	0.9969	0.9958	0.9975	0.9948	0.9980	
F	0.9955	0.9938	0.9966	0.9931	0.9971	
FRiesz	0.9982	0.9984	0.9989	0.9968	0.9987	
iFRiesz	0.9973	0.9977	0.9986	0.9966	0.9984	

Table 7: Garch Parameter