Probability Distributions for Realized Covariance Matrices

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- Derivation of own probability distributions with desirable properties (which are added to the comparsion).
- Derivation of a class of GAS models for the distributions (for which the Scores and Fisher Information matrices are required).

Realized Covariance Matrix

$$\underset{(p\times p)}{\mathbf{RC}_t} \equiv \sum_{j=1}^{m} \mathbf{r}_{j,t}\mathbf{r}_{j,t}^{\top}$$

- $\mathbf{r}_{j,t}$ is the j'th intraday return vector $(p \times 1)$ on day t.

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- $\mathbf{RC}_t$ is symmetric positive definite (s.p.d.) (if $m \geq p$).
- $\mathbf{RC}_t$ is a consistent estimator of the integrated covariance of day t.

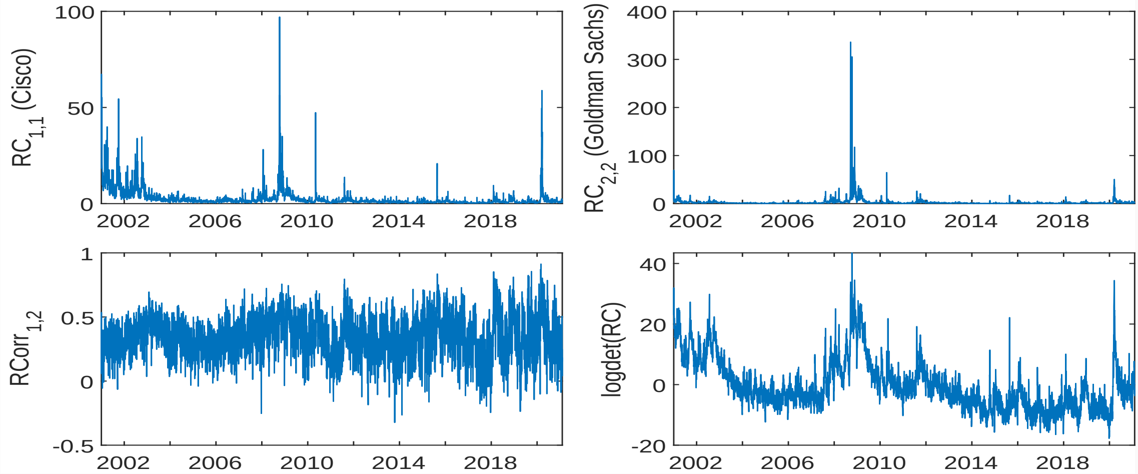
Dataset

From *Quantquote* we obtained 1-min OHLC equity data from which we constructed trading-daily RCs created from

- 5min returns with subsampling over the trading day
- for a random selection of 3, 5, 10, 25 and 50 assets
- from 02.01.2001 to 05.02.2021.

Data Plot

10 assets dataset: csco,gs,amat,bac,c,intc,jnj,jnpr,jpm,ms



Probability Distributions for RCs - Setup

$$\mathbf{RC}_t|\mathcal{F}_{t-1} \sim f(\mathbf{\Omega}_t, \mathbf{n}, \nu)$$

- f is a probability distribution on the space of s.p.d. matrices.

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- $\underset{(p\times 1)}{\mathbf{n}}$ and $\underset{(p\times 1)}{\boldsymbol{\nu}}$ are parameter vectors (degrees of freedom (dfs))
- $\mathbb{E}\left[\mathbf{RC}_t|\mathcal{F}_{t-1}\right] = \boldsymbol{\Omega}_t$.

Probability Distributions used in Literature

(Non-central)-Wishart:

Golosnoy, Gribisch and Liesenfeld (2012); Gorgi, Hansen, Janus and Koopman (2019); Yu, Li and Ng (2017)

Inverse Wishart:

Gourieroux, Jasiak and Sufana (2009); Asai and So (2013)

Matrix-F:

Opschoor, Janus, Lucas and Van Dijk (2018); Zhou, Jiang, Zhu and Li (2019)

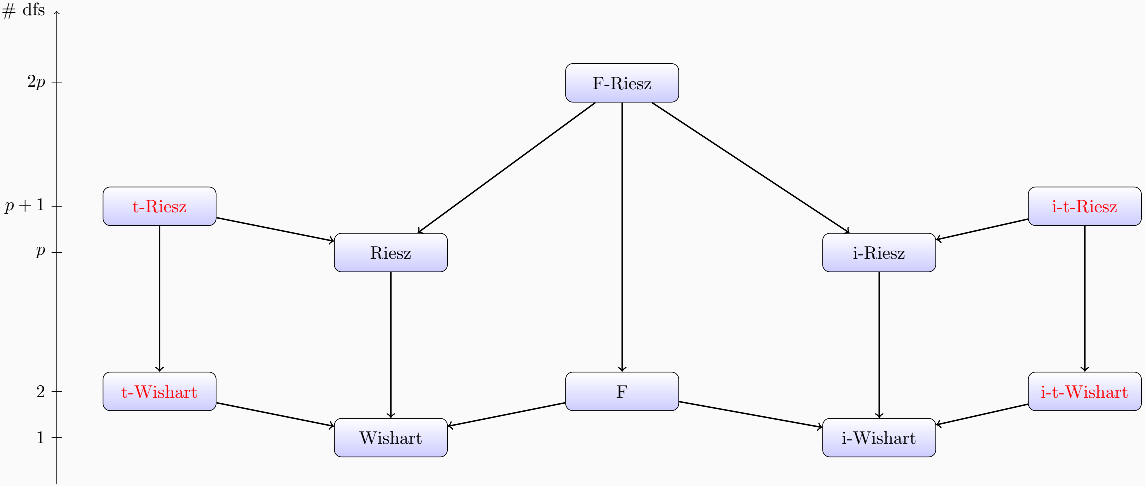
Riesz:

Gribisch and Hartkopf (2021)

Inverse Riesz and **F-Riesz**:

Blasques, Lucas, Opschoor and Rossini (2021)

Probability Distribution Relationships

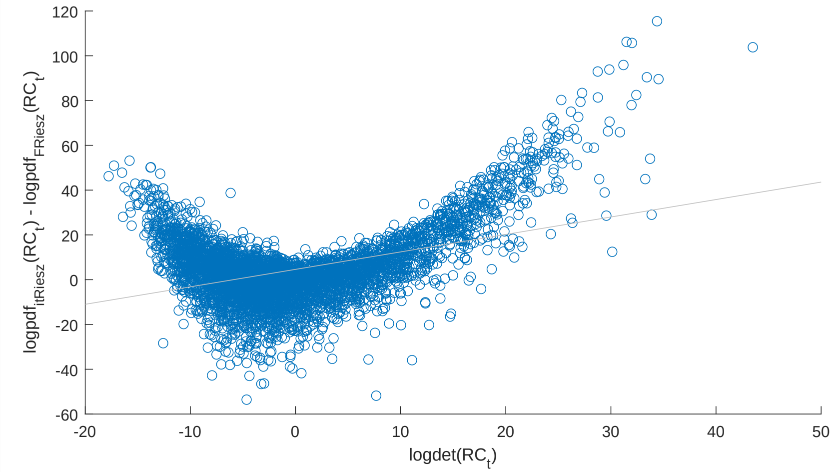


Static ($\Omega_t = \Omega$) Probability Distribution Fit - Log Likelihood Values

# Assets:	3	5	10	25	50
Wishart	-40156	-113087	-217773	-233119	807058
Riesz	-37418	-97950	-147977	66296	2013572
iWishart	-33297	-79296	-132981	129272	2017817
iRiesz	-32061	-73941	-86725	330305	2987798
F	-32960	-78546	-118624	231591	2745856
FRiesz	-25434	-51341	-8673	653323	4065553
tWishart	-24551	-59336	-26164	508370	3354653
tRiesz	-22648	-50216	6445	676273	4012244
itWishart	-24583	-55815	-23861	597007	3597630
itRiesz	-23454	-51728	13125	756666	4389476

Static ($\Omega_t = \Omega$) Probability Distribution Fit - FRiesz vs itRiesz

10 asset dataset



Generalized Autoregressive Score (GAS) Models

$$\mathbf{RC}_t | \mathcal{F}_{t-1} \sim f(\boldsymbol{\Omega}_t, \mathbf{n}, \nu)$$
$$\boldsymbol{\Omega}_t = \boldsymbol{\Phi} + \alpha \mathbf{S}_{t-1} + \beta \boldsymbol{\Omega}_{t-1},$$

$\underset{(p \times p)}{\boldsymbol{\Omega}_t}$ is s.p.d., α and β are scalars.

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$$\mathbf{S}_t = \mathrm{ivech}\left(\mathcal{I}_t^{-1}\nabla_t^\top\right),$$

where

$$\nabla_t = \frac{\partial log\ p(\mathbf{RC}_t|\boldsymbol{\Omega}_t, \mathbf{n}, \nu; \mathcal{F}_{t-1})}{\partial \mathrm{vech}\left(\boldsymbol{\Omega}_t\right)^\top}$$

and

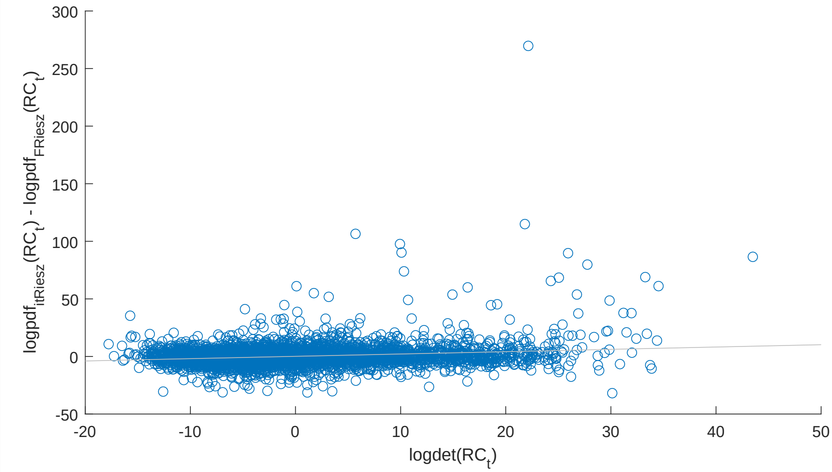
$$\mathcal{I}_t = \mathbb{E}\left[\frac{\partial^2 log\ p(\mathbf{RC}_t|\boldsymbol{\Omega}_t, \mathbf{n}, \nu; \mathcal{F}_{t-1})}{\partial \mathrm{vech}\left(\boldsymbol{\Omega}_t\right) \partial \mathrm{vech}\left(\boldsymbol{\Omega}_t\right)^\top}\right].$$

GAS Model Fit - Log Likelihood Values

# Assets:	3	5	10	25	50
Wishart	-21289	-35008	42309	782755	4328265
Riesz	-19747	-33236	52177	816447	NaN
iWishart	-16085	-24709	77840	1004169	5026229
iRiesz	-15439	-23395	84601	1036836	NaN
F	-15704	-23417	86321	1060002	5399298
FRiesz	-12331	-13753	115632	1161653	NaN
tWishart	-14975	-21448	93561	961094	4842077
tRiesz	-13971	-19583	100281	1017146	NaN
itWishart	-13493	-16529	111835	1129492	5487452
itRiesz	-12833	-14608	114988	1156506	NaN

GAS Model Fit - FRiesz vs itRiesz

10 asset dataset



GAS Model Forecasting Ability - Setting

We use the

- 10 assets dataset and
- starting from $t = 1250$ estimate the model daily with 1250 trailing observations,
- make 1-step ahead forecasts
- and evaluate the forecasting ability with two different loss functions.

data plot

GAS Model Forecasting Ability - Log Likelihood Loss (a.k.a. Log Score)

Plug $\mathbf{RC}_{t+1}$ into the in time-t forecasted log probability density function,

$$\mathit{log}\ p(\mathbf{RC}_{t+1}|\widehat{\mathbf{\Omega}_{t+1}}, \widehat{\mathbf{n}}, \widehat{\boldsymbol{\nu}}; \mathcal{F}_t)$$

to obtain a time series of losses beginning $t = 1251$.

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Distribution	Average Log-Score	DM-Teststat vs FRiesz
Wishart	13.0684	8.9490
Riesz	15.1021	8.3160
iWishart	21.6405	14.5129
iRiesz	23.0788	13.8030
F	23.2978	34.4153
FRiesz	29.2080	
tWishart	24.4700	16.8492
tRiesz	25.8310	11.9864
itWishart	28.4841	3.9231
itRiesz	29.1163	0.4708

Economic Relevance - Low Volatility Funds

Name	Symbol
Voya US High Dividend Low Volatility	VHDIX
Wells Fargo Low Volatility US Eq R6	WLVJX
SEI Tax-Managed Mgd Volatility F (SIMT)	TMMAX
SEI US Managed Volatility A (SIIT)	SVYAX
SEI US Managed Volatility F (SIMT)	SVOAX
Janus Henderson US Managed Volatility I	JRSIX
LSV US Managed Volatility Institutional	LSVMX
MFS Low Volatility Equity R6	MLVTX
Invesco Low Volatility Equity Yield R5	SCIUX
Invesco US Managed Volatility R6	USMVX
Fidelity® SAI US LowVolatility Idx	FSUVX
Fidelity® US Low Volatility Equity	FULVX
BMO Low Volatility Equity I	MLVEX

Economic Relevance - GAS Model Forecasting

If we want to **minimize** the realized portfolio variance

$$\mathbf{RV}_t^{portf} = \mathbf{w}_t^\top \mathbf{RC}_t \mathbf{w}_t$$

for a given set of assets, then we must set portfolio weights as

$$\mathbf{w}_t^* = \frac{\mathbf{RC}_t^{-1} \mathbf{1}}{\mathbf{1}^\top \mathbf{RC}_t^{-1} \mathbf{1}}.$$

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- $\mathbf{w}_t^*$ is random and depends on distribution for $\mathbf{RC}_t$.

Economic Relevance - GAS Model Forecasting

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$$\mathbf{w}_t^* = \frac{\mathbf{RC}_t^{-1} \, \mathbf{1}}{\mathbf{1}^\top \, \mathbf{RC}_t^{-1} \, \mathbf{1}}.$$

- $\mathbf{w}_t^*$ is random and depends on distribution for $\mathbf{RC}_t$.
- $\mathbb{E}\left[\mathbf{w}_t^*\right]$ can easily be simulated, given the assumed distribution, $\boldsymbol{\Omega}_t$, $\mathbf{n}$ and $\boldsymbol{\nu}$.

GAS Model Forecasting Ability - GMRVP

1. Predict $\mathbf{w}_{t+1}^*$ with simulated $\mathbb{E}\left[\mathbf{w}_t^*\right]$,
2. then calculated the realized portfolio variance $\left(\mathbf{w}_{t+1}^*\right)^\top \mathbf{RC}_{t+1}\,\mathbf{w}_{t+1}^*$

to obtain a time series of losses beginning $t = 1251$.

GAS Model Forecasting Ability - GMRVP

1. Predict $\mathbf{w}^*_{t+1}$ with simulated $\mathbb{E}\left[\mathbf{w}^*_t\right]$,
2. then calculated the realized portfolio variance $\left(\mathbf{w}^*_{t+1}\right)^\top \mathbf{RC}_{t+1}\, \mathbf{w}^*_{t+1}$

to obtain a time series of losses beginning $t = 1251$.

Distribution	Average Realized Portfolio Variance	DM-Teststat vs tWishart
Wishart	.6285	2.9047
Riesz	.6298	2.5485
iWishart	.6393	2.7885
iRiesz	.6376	3.0871
F	.6236	2.6119
FRiesz	.6295	2.7809
tWishart	.6219	
tRiesz	.6220	0.3064
itWishart	.6261	3.9086
itRiesz	.6261	3.6735

Economic Relevance - GAS Model Forecasting

Do you have ideas?